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Coordinating for Clicks: Learning in Multi-Agent Information Asymmetric Cascading Bandits

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Abstract

We formulate a decentralized, cooperative multi-agent bandit framework, studied in Chang et al. (2022), applied to the stochastic partial-monitoring cascading bandit problem first introduced in Kveton et al. (2015a). The reward in each round depends on the joint-ranking "cascade" collectively taken by all learning agents. The objective is shared, but to make the coordination problem more challenging, we contend with three variants of information asymmetry: action asymmetry, where the overall joint-ranking is unobservable to all agents but the feedback received is common; reward asymmetry, where the overall ranking is observable, but feedback received by each agent is i.i.d.; and that with both action and reward asymmetry. For the first setting, we propose `mCascadeUCB`, and for the second setting, we propose `mCascadeUCB-Intervals`, with both algorithms achieving $O(\log T)$ gap-dependent regret within their respective settings. For the last setting, we propose `mCascadeDSEE`, which achieves close to $O(\log T)$ gap-independent regret. We demonstrate our algorithms with experimental results.

Keywords: Multi-Armed Bandit (MAB), Cascading, Multiplayer, Information Asymmetry, Reinforcement Learning

1 Introduction

In applications from search engine optimization to network routing to user recommendation, the *cascade model* is a prominent tool used to describe user behavior or instances of partial feedback. In the early twenty-first century, user behavior research examined how characteristics of implicit user feedback and engagement—click-through rate, dwell times, query chains, etc.—could be modeled and employed in recommendation or web search algorithms to improve performance by up to thirty-one percent relative to algorithms without this data (Agichtein et al. (2006), Radlinski and Joachims (2005)). Craswell et al. (2008) introduced the first notion of the cascade model that sought to account for user "position bias" in search rankings.

In essence, the model assumes a user that examines *from first to last* an ordered ranking or list of items. The user endows each item with an attraction probability, which is the likelihood the user is "attracted" by the item. An important assumption is that the user will continue to examine the list until they are attracted by an item or reach the end of the list. Additionally, if an item has attracted the user (often interpreted as a "click"), the user stops and does not examine the remaining items. Thus, this model incorporates position bias, in that higher-ranked items are more likely to be observed or clicked by users.

Previous literature has extended this model to a reinforcement learning bandit problem. Kveton et al. (2015a) presented the first extension of the cascade model to a stochastic bandit setting. We rebuild the technical details in section 2.1, but primarily, they consider a learning agent whose goal is to minimize regret by recommending the top K items while receiving cascade "click" feedback. In particular, they develop the first algorithms for the cascading setting, `CascadeUCB1` and `CascadeKL-UCB`, and prove gap-dependent regret upper bounds on the order of $O(\log T)$. Many related works to the cascading setting have since been studied, which we examine further in section 1.

In a different vein, the study of multi-agent multi-armed bandits (MAMAB) has progressed in recent years, motivated by applications in areas such as cognitive radio networks (Boursier and Perchet (2022)). In particular, one of the first frameworks for a decentralized, cooperative multi-agent setting for general multi-armed bandits is presented in Chang et al. (2022) and Chang and Lu (2023). They introduce the notion of all individual players' actions comprising a collective *joint-action* with its own distinct reward distribution. Thus, the common goal for all players is to minimize collective regret by pulling the most optimal joint-arm.

Furthermore, because of multi-agent dynamics, Chang et al. (2022) also introduce forms of information asymmetry between the players into their settings. First, that in which players are unable to observe the actions taken by other players, but receive identical rewards from the joint-action (action asymmetry); second, that in which players receive i.i.d. copies of the reward, but they can observe each others' actions (reward asymmetry); and finally, where players cannot observe each other's actions *and* receive i.i.d. copies of the reward (both action and reward asymmetry).

Our contribution. We make four contributions. First, we formulate a novel extension of the stochastic cascading bandit introduced in Kveton et al. (2015a) to a multi-agent setting, equipped with *joint-items* and *joint-rankings*. Previous work in this intersection (Yang et al. (2024)) has explored multi-agent systems with communication between players facilitated by a central server, but in our paper, our environment is decentralized and we allow no explicit communication between players during play. The setting and problem setup are expanded upon further in section 2. Second, we increase the challenge of player coordination by considering three problems involving information asymmetries inspired by Chang et al. (2022), proposing algorithms for each. They are as follows:

- **Problem A: Action Asymmetry** We propose an ordering on the joint-items that allows the players to coordinate even without having to observe the actions of the other players. See Algorithm 1, `mCascadeUCB`.
- **Problem B: Reward Asymmetry** We propose an interval-based approach that allows players to successively eliminate suboptimal joint-items despite receiving different rewards. See Algorithm 2, `mCascadeUCB-Intervals`.
- **Problem C: Action and Reward Asymmetry** We propose a variant of the explore-then-commit algorithm that achieves nearly optimal regret. See Algorithm 3, `mCascadeDSEE`.

Further descriptions of the algorithms can be found in section 3. Third, we prove gap-dependent regret upper bounds, notably for `mCascadeUCB-Intervals` in the reward-asymmetric setting, on the order of $O(\log T)$, and a gap-independent regret upper bound for `mCascadeDSEE` in the action and reward-asymmetric setting that achieves close to log regret. The details can be found in section 4. Finally, we showcase the performance of our algorithms with several experiments, the results of which can be found in section 5.

Related Works. Various advancements to the canonical single-agent cascading bandit in Kveton et al. (2015a) have been made. Thompson Sampling algorithms such as TS-Cascade (Cheung et al. (2019), Zhong et al. (2021)) and variance-aware confidence sets derived from Bernstein and Chernoff bounds match previous algorithms' performance in the cascading setting (Vial et al. (2022)). Zong et al. (2016) formulate a *linear* cascading bandit, proposing algorithms that improve the regret dependence on the number of items, L, via the assumption that item attraction probabilities can be approximated via an environment parameter vector.

Other popular cascading bandit variants are contextual cascading bandits (Li et al. (2016), Wang (2021), Choi et al. (2024)), where information or context vectors on the items are observed; best-arm/top K item identification (Zhong et al. (2020)); non-stationary settings with abruptly changing attraction probabilities (Li and De Rijke (2019)); and even an extension to a cascading Markov decision process, where transitions to more auspicious states must be considered (Du et al. (2024)). As touched upon earlier, Yang et al. (2024) consider a *federated* contextual cascading bandit, where each user is *individually* served by an agent, in contrast to our setting where agents' actions comprise a *joint-ranking* for the user(s). Agent communication is also facilitated through a central server, while our decentralized setting features no communication during play and various information asymmetries, giving rise to a distinctly complex cooperation problem.

Modifications to the cascade model have been implemented into bandits as well. One is user and topic *clustering*, with frameworks featuring item "topics" and user "interests" with correlated attraction probabilities (Combes et al. (2015)) or even an underlying cluster graph where an edge denotes user similarity which can be learned while incurring $O(\sqrt{T})$ regret with the CLUB-Cascade algorithm (Li and Zhang (2018)). Katariya et al. (2016)

extend a similar "Dependent Click Model" (DCM) to a bandit setting, which adds an aspect of "user satisfaction" that just a "click" might not imply. Mansoury et al. (2024) implement "exposure bias"—where more "useful" recommendations are pushed down by those consistently recommended—by discounting rewards to items clicked higher in the ranking. An item's attraction probability could depend on its variation from previous items' topics, adding "diversity" (Li et al. (2020), Becker et al. (2007)). Item "abandonment probabilities" (Cao et al. (2019)) remove the assumption that users will always continue examining. For a final exhibit, Wang et al. (2024) incorporate delayed click feedback.

On the other hand, our work intersects with the growing literature on the multi-agent or cooperative bandits problem first introduced by Awerbuch and Kleinberg (2008). In this setting, M players have a collective goal of determining the best action, and often, a graph represents the communication framework between players. Subsequent algorithms, such as ϵ-greedy variations (Szorenyi et al. (2013), Jin et al. (2023)), gossip UCB (Landgren et al. (2016), Martínez-Rubio et al. (2019)), or leader-follower DPE1 (Wang et al. (2020)) have been proposed. Other lines of multi-agent works involve players only observing the rewards of players within a neighboring distance (Cesa-Bianchi et al. (2016)) or realizing heterogeneous rewards while communicating information via a graph network (Xu and Klabjan (2024)). Some multi-agent settings involve asynchronous actions, where only a subset of players are active at any time (Bonnefoi et al. (2017), Cesa-Bianchi et al. (2020)), or voting systems to select the best action in a shared network set (Shahrampour et al. (2017)). Dubey et al. (2020) consider a group of communicating agents selecting actions from their *individual* action sets, with the goal of minimizing total group regret.

The decentralized, limited communication, multi-agent setting introduced in Chang et al. (2022) has been explored further as well, with cooperative multi-agent reinforcement learning algorithms following (Kao (2022), Mao et al. (2021), Mao et al. (2022)) that do not require communication among players or a central server during learning. Furthermore, information asymmetric RL settings have been explored, such as the leader-follower games in Kao et al. (2022) where only the follower observes the action of the leader while realizing the same reward, or extensions of information asymmetry to multi-agent contextual bandits in Chang and Lu (2024). To the best of our knowledge, information asymmetry in a multi-agent cascading bandit setting has yet to be researched.

Various areas call for answers within the intersection of cascading models and multi-agent reinforcement learning. The user-interaction and click framework (Craswell et al. (2008), Kveton et al. (2015a)) we adopt in this paper translates to a multi-agent problem when we consider complex, multi-faceted items or recommendations that involve the collaboration of multiple parties (e.g. content, thumbnails, and titles for YouTube videos; scripts and graphic design for advertisements, etc.). Another area that might find utility, particularly from the information asymmetry we study, is network routing. Kveton et al. (2015b) take this framework in single-agent cascading bandits, where the Bernoulli feedback is instead interpreted as points in a chosen network route that are on or down. *Multi-agent* reinforcement learning in particular has potential in multi-agent routing (Yamin and Permuter (2024), Zeng et al. (2020), Mammeri (2019)). Physical or cost constraints may hinder communication or a central server for networks, calling for no-communication solutions. Additionally, a network path's efficiency or latency is highly dependent on the (potentially-unobservable) actions of other stations, and can vary from station to station, analogous to the joint-actions and information asymmetry we study. This leaves stations having to coordinate to maximize overall network performance—a cooperative multi-agent problem.

2 Preliminaries and Problem Statements

2.1 Single-Agent Cascading Bandits

First, we set up the standard cascade model and its inspired bandit. For any round t up to the horizon T, a learning agent plays a *ranking* or action $A = (a_1, \ldots, a_K)$ which consists of an ordered list of K items chosen from the agent's ground set of items with cardinality L. A user then examines the ranking from the first to the last item. The model (or user) endows each item $e \in E$ with an *attraction probability*, $\bar{w}(e) \in [0, 1]$, which is the likelihood item e "attracts" the user, and the user "clicks". This probability is assumed to be independent of the other items.

As previously introduced, an important assumption of the cascade bandit is that once the user has clicked one item $a_k \in A$, the user stops and does not examine the remaining items in A. In other words, items $a_1, \ldots, a_{k-1}$ are considered to be unattractive to the user for that A_t, while the attractiveness of items $a_{k+1}, \ldots, a_K$ are *unobserved.* The reward for the agent takes on a value of 1 if the user clicked on any item in the agent's ranking, and 0 if no items were clicked. Ultimately, for any ranking A, a user can make at most one click and the agent's goal is to maximize the likelihood the user clicks on an item, which equates to choosing the $A^* \in \Pi_K(E)$ consisting of the K most attractive items such that $1 - \Pi_{i=1}^{K}(1 - \bar{w}(a_i))$ is maximized.

2.2 Multi-Agent Extension of Cascading Bandits

Here we extend the single-agent cascading bandit problem and formulate the decentralized multi-agent setting by introducing a new framework allowing for *joint actions* by multiple agents.

Let M be the number of players and E^i denote the set of actions player i has access to. At the start of every round, each player picks K actions from their own action set. This results in a *joint action or ranking* taken by these players which we denote using a tuple $A = (\boldsymbol{a_1}, .., \boldsymbol{a_K})$ of K *joint items.* Each joint item $\boldsymbol{a}$ is some vector $(a^1, ..., a^M)$ where $a^i \in E^i$. [1] Additionally, we make the assumption that for all players i, $\left|E^i\right| = L$. Thus, the number of distinct joint items by M players is L^M and the number of joint rankings with K distinct items is $\frac{(L^M)!}{(L^M-K)!}$. In standard no-communication multi-agent bandits, the players are allowed to agree on a strategy prior to the learning process; however, once the learning begins they cannot explicitly communicate. Therefore, we have the following modification from the single-agent setting.

Remark 1 *In the single agent setting, at every step, the player recommends K out of L items for the user to click on. For the multiplayer setting, this process will occur via players taking their own actions K times in an ordering of their choice. When the user clicks on a*

[1] To avoid notation confusion, for a given item, a subscript indicates the *ranking position* of that *joint item* which is recommended, while a superscript indicates the *player* whom an item is from.

joint-item the players will know the place number of the item, but not the joint item itself. e.g. they will know it's the 1st, 2nd, 3rd, etc. item that was recommended.

We let $C_t \in \{1, \ldots, K, \infty\}$ be this feedback, i.e. the item that was clicked by the user, where $C_t = \infty$ is equivalent to no click occurring. This remark is necessary for the information asymmetry in actions (see problem A below) to be meaningful.

A subtle point to mention is with how the bandit is structured, the multiplayer setting now creates a nontrivial possibility of *repeated* items. Recall in the single-player setting, an agent takes an action $A \in \Pi_K(E)$, which eliminates the possibility of item repetition. However, in the multiplayer setting, players can *intend* to play certain permutations of joint-items, but because they only control *their respective item*, this can result in a joint-ranking with joint-item repetition. While repetition ostensibly seems "legal" in the sense of a cascade model, it clashes with the intuition of applications such as a recommendation system. Fortunately, the setup in the following paragraph helps punish this repetition.

Let $B(p)$ denote the Bernoulli distribution with parameter p. Our user assigns a Bernoulli distribution to each of the L^M joint items which remains fixed across time. For a joint item, an outcome of 1 is interpreted as attractive (where the user will click on the item) and 0 as unattractive (where the user will skip over that item). Define $\bar{w}(\boldsymbol{e})$ to be the true attraction probability for joint item $\boldsymbol{e}$. For each round t, let $\mathbf{w}_t \in \{0,1\}^{L^M}$ be drawn from the joint distribution $\prod_{\boldsymbol{e}} B(\bar{w}(\boldsymbol{e}))$, which encodes the attractiveness of all joint items for the user at time t. In other words, if for some joint-item $\boldsymbol{e}$, $\mathbf{w}_t(\boldsymbol{e}) = 1$, then the user, given that they *observe* item $\boldsymbol{e}$, would be attracted by or click on item $\boldsymbol{e}$ at time t. Note that while $\bar{\boldsymbol{w}}$ stays constant throughout all rounds, $\mathbf{w}_t$ is instantiated for round t only.

Remark 2 *The drawing of $\mathbf{w}_t$ for each round t punishes any instance of repeated joint-items, as the probability of the user being attracted to each of the repeated items is no longer independent; a user will assign 1 or 0 equally to all instances of that item for the duration of round t. Thus, in maximizing the probability a joint-ranking gets a click, it is in the players' best interests to always have K distinct joint-items.*

Avoiding this repetition is not easy, particularly in "collision-prone" multiplayer problems. Ultimately, we seek to contend with emergent complexities of a multi-agent setting by considering potential forms of *information asymmetry* present. The individual problems we study are expanded upon in the following sections.

Problem A *Asymmetry in Actions.* In this setting, each player is unable to observe the other players' actions, but they observe the same reward. In other words, the overall joint-ranking $\boldsymbol{A}_t$ is unobservable to all players, but the item weights drawn by the user $\mathbf{w}_t$ apply to all players.

Problem B Asymmetry in Rewards. In this setting, each player can observe the other player's actions, but they each obtain their own i.i.d. realization of the reward. They cannot observe other player's realizations.

Problem C Asymmetry in both Actions and Rewards. In this setting, each player is unable to observe the other player's actions, and they obtain their own i.i.d realization of the reward.

We define the reward function f for some player i on any round t as follows:

$$f(\mathbf{A}_t, \mathbf{w}_t) = 1 - \prod_{k=1}^{K} (1 - \mathbf{w}_t(\boldsymbol{a}_k)) \tag{1}$$

We evaluate a learning policy by *all* the players by its *expected regret*, defined as:

$$R_T = \mathbb{E}\left[\sum_{t=1}^{T} f(\mathbf{A}^*, \mathbf{w}_t) - f(\mathbf{A}_t, \mathbf{w}_t)\right] \tag{2}$$

where $\mathbf{A}^*$ is an optimal joint-ranking consisting of the K items with the highest true attraction probabilities.

Note that from Remark 1 for Problem A, the players *all* receive a reward of 1 if at least one joint-item in $\boldsymbol{A}_t$ is attractive to the user at time t, as each player observes the same ranking-position of the item clicked, C_t, because $\mathbf{w}_t$ is shared. Also note that because of the reward asymmetry in Problems B and C, each player receives their own reward, but since the rewards are i.i.d., they will experience the same regret as regret is calculated in *expectation.*

3 Main Results and Algorithms

3.1 Problem A: Information Asymmetry in Actions

Recall in Problem A, players cannot observe the exact joint action taken. Incoordination arises easily, as joint-items played can differ from those intended. Consider for a given position, player 1 wants to play joint item (a, b) and player 2 wants to play (c, d). Since each player only controls their respective item, the resulting joint item is (a, d). Not only could this lead to repetition of joint-items, decreasing the likelihood of the ranking getting a click, but if this joint-item is observed, both players would record the observation and feedback for the wrong joint-item. Thus, players must somehow coordinate effectively and *infer* the actions of all other players. To solve this challenge of information asymmetry in actions, we introduce the `mCascadeUCB` algorithm. The psuedocode is in Algorithm 1. `mCascadeUCB` is a multiplayer adaptation of `CascadeUCB1` (Kveton et al., 2015).

Players first estimate the user's attraction probabilities for each joint-item on each round by calculating their upper-confidence bounds (UCB). As in `CascadeUCB1`, for each item $\boldsymbol{e}$, the UCB at time t is given by:

$$\mathrm{UCB}_t(\boldsymbol{e}) = \begin{cases} \infty & \text{if } n_{t-1}(\boldsymbol{e}) = 0 \\ \widehat{\mathbf{w}}_{n_{t-1}(\boldsymbol{e})}(\boldsymbol{e}) + c_{n_{t-1}(\boldsymbol{e})} & \text{otherwise.} \end{cases} \tag{3}$$

where $\widehat{\mathbf{w}}_s(e)$ is the average of s observed weights of joint-item $\boldsymbol{e}$, $n_t(\boldsymbol{e})$ is the number of times that item e is observed after t rounds, and:

$$c_s = \sqrt{\frac{1.5 \log T}{s}} \tag{4}$$

is the radius of a confidence interval around $\widehat{\mathbf{w}}_s(e)$ after t steps such that $\bar{w}(\boldsymbol{e}) \in [\widehat{\mathbf{w}}_s(\boldsymbol{e}) - c_s, \widehat{\mathbf{w}}_s(\boldsymbol{e}) + c_s]$ holds with high probability.

Each player then plays their own K items that correspond to the K joint-items with the highest UCBs.[2] Here arises two concerns: 1) what should a player do when two or more joint-items' UCBs are equal, and 2) whether UCB indices for items differ between players, leading to players intending for different joint-actions. For 1), ties in the single-player UCB algorithm were settled arbitrarily, however, this fails in the multiplayer setting. So to address this issue, we define an *order relation* on $\boldsymbol{E}$, which players can agree upon prior to learning and follow throughout the game. We define the joint items order relation as follows:

Definition 3 *Let M be the number of players and let $\boldsymbol{a} = (a^1, \ldots, a^M), \boldsymbol{b} = (b^1, \ldots, b^M)$ be two joint items in $\boldsymbol{E}$. We say $\boldsymbol{a} < \boldsymbol{b}$ if and only if there exists an $n \in \{1, 2, \ldots, M\}$ such that for all $i < n, a_i = b_i$ and $a_n < b_n$.*

In `mCascadeUCB-A`, players then settle UCB ties between joint-items choosing the *lesser* joint-items according to this relation. This ensures all players choose the same K items, at least for the first round. This coordination will continue for all rounds, because for Problem A, C_t is the same for all players. Thus, all players will update UCBs over $\boldsymbol{E}$ identically, resolving 2), and ensuring for any round, all players can infer what joint-ranking will be played.

We show that `mCascadeUCB` achieves a similar upper bound on the expected T-step regret as the single-agent `CascadeUCB1` algorithm. We first assume without loss of generality that the joint-items in $\boldsymbol{E}$ are sorted by decreasing attraction probabilities so that $\boldsymbol{A}^* = (\boldsymbol{1}, \boldsymbol{2}, \ldots, \boldsymbol{K})$ is the optimal ranking, consisting of the K *optimal joint-items.* Thus, joint-items $\boldsymbol{K+1}, \ldots, \boldsymbol{L^M}$, are called *sub-optimal.* The *sub-optimality gap* between an optimal item $\boldsymbol{e}^*$ and a suboptimal item $\boldsymbol{e}$ is defined to be the following:

$$\Delta_{\boldsymbol{e},\boldsymbol{e}^*} := \bar{w}(\boldsymbol{e}^*) - \bar{w}(\boldsymbol{e})$$

We make these assumptions again in later proofs of Problems B and C.

Theorem 4 *If each player uses* `mCascadeUCB-A` *in the setting of Problem A, then the expected T-step regret of* `mCascadeUCB-A` *is bounded as:*

$$R_T \leq \sum_{\boldsymbol{e}=\boldsymbol{K+1}}^{\boldsymbol{L^M}} \frac{12}{\Delta_{\boldsymbol{e},\boldsymbol{K}}} \log T + \frac{\pi^2}{3} L^M.$$

This upper bound is a direct corollary from Theorem 2 as proved in Kveton et al. (2015a). As using `mCascadeUCB-A` essentially removes all action information asymmetry between the players and the players' UCB indices are identical throughout the play, Problem A becomes analogous to a single player cascading bandit, with the main distinction being there are now L^M total joint-items and K optimal joint-items with the highest attraction probabilities comprising the optimal ranking.

[2] Note that the reward is the same for all permutations of a joint-ranking, but as shown in the psuedocode for `mCascadeUCB-A`, we arrange these K items from lowest to highest UCB to increase the likelihood of more items being observed. This often is seen as a shortcoming of the cascade model, as intuition would want the most attractive item to the user ranked first.

Algorithm 1: `mCascadeUCB`

```
Each player $i$ has a ground set $E^i$ consisting of $L$ items. Denote $\boldsymbol{e}$ to be a joint item
  and $\boldsymbol{E} = \Pi_M(E^i)$ to be the set of all joint items by $M$ players. Players will agree to
  an ordering of $\boldsymbol{E}$ held constant during the entire game.
$\forall \boldsymbol{e} \in \boldsymbol{E}$, initialize $\widehat{\mathbf{w}}_0(\boldsymbol{e}) = 0$ and $n_0(\boldsymbol{e}) = 0$.
for $t = 1, \dots, T$ do
    $\forall \boldsymbol{e} \in \boldsymbol{E}$, compute $\mathrm{UCB}_t(\boldsymbol{e})$ (see 3).

    // Joint ranking selection
    Each player $i$ considers the $K$ joint items with the largest UCBs. If there exist
      ties, preference goes to the lesser joint item(s) indicated by the order relation
      specified in Definition 3. These $K$ items are then sorted from lowest UCB to
      highest UCB. Let $\boldsymbol{a_1}, \dots, \boldsymbol{a_K}$ be these $K$ joint items after this selection and
      sorting.
    Each player $i$ then selects their respective individual $K$ items from $E_i$ in the
      order that makes up this order of $K$ joint actions.
    $(\boldsymbol{a_1}, \dots, \boldsymbol{a_K}) \to \boldsymbol{A}_t$
    Each player observes same click $C_t \in \{1, \dots, K, \infty\}$.

    // All players update relevant statistics
    $\forall \boldsymbol{e} \in E, \boldsymbol{n}_t(\boldsymbol{e}) = \boldsymbol{n}_{t-1}(\boldsymbol{e})$
    forall $k \in \{1, 2, \dots, C_t\}$ do
        $\boldsymbol{a_k} \to \boldsymbol{e}$
        $\boldsymbol{n}_t(\boldsymbol{e}) + 1 \to \boldsymbol{n}_t(\boldsymbol{e})$
        $\dfrac{\widehat{\mathbf{w}}_{t-1}(\boldsymbol{e})\boldsymbol{n}_{t-1}(\boldsymbol{e}) + \mathbb{1}\{k = \mathbf{C}_t\}}{\boldsymbol{n}_t(\boldsymbol{e})} \to \widehat{\mathbf{w}}_t(\boldsymbol{e})$
    end
end
```

3.2 Problem B: Information Asymmetry in Rewards

In Problem A, since all players observe the same reward outcome (i.e., the same click), they maintain identical updates for each joint item's empirical mean and confidence interval over time. In other words, each player has the same ordering of the joint items at every step. A fixed ordering relation of joint items, used only to break ties in UCB values, thus suffices to ensure that all players choose the same top K items.

In Problem B, however, each player sees their own i.i.d. reward realization for any recommended item and will consequently form different empirical means for the same joint item. As a result, even for the same joint item, players will no longer share a common set of UCB values. Applying `mCascadeUCB` could give rise to miscoordination — where one player aims for one joint arm, but another player aims for another joint arm, thus resulting in taking a completely different joint arm (see Figure 1 in Chang et al. (2022) for an illustration in the standard MAB problem that using UCB indices alone results in linear regret for Problem B).

Thus for Problem B, we propose an intervals-based algorithm, `mCascadeUCB-Intervals`, the psuedocode of which is in Algorithm 2. Recall that the familiar UCB index ensures the true mean of an item lies below it with high probability, but by also subtracting the error term, $c_{n_{t-1}}(\boldsymbol{e})$, from the empirical mean, $\widehat{\mathbf{w}}_{n_{t-1}(\boldsymbol{e})}(\boldsymbol{e})$, we obtain the *lower confidence bound* (LCB), which with high probability, lies below the true mean for a joint-item. Together, the *UCB interval* contains the true attraction probability for a joint-item with high probability.

UCB intervals are powerful as if two intervals do not overlap (i.e. are disjoint), then with high probability, we can say the item corresponding to the lower interval has a worse click probability than the item corresponding to the higher interval. Also note that by the error term $c_{n_{t-1}}(\boldsymbol{e})$ calculation, these intervals will be monotonically shrinking as items get observed repeatedly. Thus, in the case of cascading bandits specifically, where players need to recommend the top K joint-items (out of L^M), players can instead work to *eliminate* suboptimal items that are *not* in the top K, adopting a quasi-"innocent until proven guilty" strategy.

Concretely, in `mCascadeUCB-Intervals`, each player keeps track of their own *desired set*, initialized to contain all joint-items. Having agreed upon an ordering of the items before play, the players then *cycle* through the items *within their desired sets* for each round, i.e. $\boldsymbol{A}_1 = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_K), \boldsymbol{a}_2 = (\boldsymbol{a}_2, \cdots, \boldsymbol{a}_{K+1}), \cdots, \boldsymbol{A}_{L^M} = (\boldsymbol{a}_{L^M}, \boldsymbol{a}_1, \cdots), \cdots$. This idea of disjoint confidence intervals is crucial, as at any time, if *any* player observes an item whose UCB interval is disjoint and below K other items, once that joint-item is meant to be played in any joint-ranking, they *sabotage*. Instead of following the agreed-upon order, they play a different individual item, which leads to a different overall joint-item. As players can observe each others' *actions*, this signals (without explicit communication) for all players to remove this item from their desired sets. Thus, using intervals is essential to avoid incoordination, as players can reliably identify and *agree* upon the best K actions, even when their individual reward observations differ.

We show that using `mCascadeUCB-Intervals` for Problem B achieves $O(\log T)$ gap-dependent regret.

Algorithm 2: `mCascadeUCB-Intervals`

```
1  Players will agree to an ordering of $\boldsymbol{E}$ held constant during the entire game.
2  For each player $i$, initialize their desired set $D = \boldsymbol{E}$, and $\forall \boldsymbol{e} \in \boldsymbol{E}$, initialize $\widehat{\mathbf{w}}_0(\boldsymbol{e}) = 0$
     and $n_0(\boldsymbol{e}) = 0$.
3  for $t = 1, \dots, T$ do
4      for each player $i \in \{1, \dots, M\}$ do
5          // Compute UCB and LCB
6          forall $\boldsymbol{e} \in D$ do
7              $\mathrm{UCB}_t(\boldsymbol{e}) = \widehat{w}_{n_{t-1}(\boldsymbol{e})} + c_{t-1,n_{t-1}(\boldsymbol{e})}$
8              $\mathrm{LCB}_t(\boldsymbol{e}) = \widehat{w}_{n_{t-1}(\boldsymbol{e})} - c_{t-1,n_{t-1}(\boldsymbol{e})}$
9          end
10         Consider the next recommendation $\boldsymbol{A}'_t$ inside $D$ in accordance to the order
             (see step 23)
11         // Update desired set
12         for each recommendation $\boldsymbol{a}'_k \in \boldsymbol{A}'_t$ do
13             if Player $i$ observes $K$ other joint arms $\mathbf{e} \in D$ satisfying
                 $UCB_t(\boldsymbol{a}_k) < LCB_t(\boldsymbol{e})$ then
14                 Player $i$ pulls the arm not equal to $\boldsymbol{a}'_k[i]$
15             end
16             else
17                 Player $i$ pulls the arm equal to $\boldsymbol{a}'_k[i]$
18             end
19         end
20         $(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_K) \rightarrow \boldsymbol{A}_t$
21         for $k = 1, ...., K$ do
22             Observe $k$th joint item of $\boldsymbol{A}_t$. if $\boldsymbol{a}_k \neq \boldsymbol{a}'_k$ then
23                 remove $\boldsymbol{a}'_k$ from $D$. For future rounds, consider the arm after $\boldsymbol{a}'_k$ (not
                     $\boldsymbol{a}_k$), according to the ordering.
24             end
25         end
26         Each player observes their own click $C_t \in \{1, \dots, K, \infty\}$.
27         // Update relevant statistics
28         $\forall \boldsymbol{e} \in E, \boldsymbol{n}_t(\boldsymbol{e}) = \boldsymbol{n}_{t-1}(\boldsymbol{e})$
29         forall $k \in \{1, 2, \dots, C_t\}$ do
30             $\boldsymbol{a_k} \rightarrow \boldsymbol{e}$
31             $\boldsymbol{n}_t(\boldsymbol{e}) + 1 \rightarrow \boldsymbol{n}_t(\boldsymbol{e})$
32             $\dfrac{\widehat{\mathbf{w}}_{t-1}(\boldsymbol{e})\boldsymbol{n}_{t-1}(\boldsymbol{e}) + \mathbb{1}\{k = \mathbf{C}_t\}}{\boldsymbol{n}_t(\boldsymbol{e})} \rightarrow \widehat{\mathbf{w}}_t(\boldsymbol{e})$
33         end
34     end
35 end
```

Theorem 5 *If each player uses* `mCascadeUCB-Intervals` *in the setting of Problem B, then the expected T-step regret of* `mCascadeUCB-Intervals` *is bounded as:*

$$R_T \leq \sum_{\boldsymbol{e}=\boldsymbol{K}+\mathbf{1}}^{\boldsymbol{L}^{\boldsymbol{M}}} \frac{12 + 48K + 48\sqrt{K}}{\Delta_{\boldsymbol{e},\boldsymbol{K}}} \log T + \left(\frac{\pi^2}{3}M + 2\right) L^M.$$

The full proof is in Section 4. We first decompose the regret using the event where all items' true means lie within their intervals, the complement of which happens with low probability. Under this event, we then bound the number of observations of suboptimal items, however, using UCB intervals means resulting inequalities include error terms for both the suboptimal item and an optimal item. This issue is alleviated by Lemma 8, which lower bounds the number of observations for any optimal item for any round $t > 2L^M$, limiting the radius of its UCB interval.

3.3 Problem C

In Problem B, players can observe each other's actions, so despite receiving i.i.d. rewards, players can maintain the same desired set by following a fixed joint-item ordering relation and eliminating joint-items when at least one player observes interval disjointness. However, the algorithm `mCascadeUCB-Intervals` fails in Problem C because players can not observe other players' actions or rewards, making it impossible to coordinate in the same way.

Since each player must rely solely on their own i.i.d. observations in Problem C, we propose the `mCascadeDSEE` algorithm, which follows a structured exploration and exploitation schedule. The players decide on an ordering and choose a monotonic function $K(\lambda)$. In the λ-th exploration phase (starting from $\lambda = 1$), each joint-item $\boldsymbol{e}$ is ranked first $K(\lambda)$ times, to ensure feedback for all items in the cascading setting. Afterward, each player commits to the top K items (arranged by the ordering) with the highest empirical means until the next power of 2. The function $K(\lambda)$ is chosen to tend to infinity so that more samples are collected in later epochs as the exploitation phases grow exponentially in length.

Although players may initially converge to different M-tuple optimal joint-items due to i.i.d. rewards, the probability of mistakes decreases rapidly as λ increases. Thus, with high probability, each player eventually identifies and commits to the true top K joint-items, forming the M-tuple optimal recommendation, $\boldsymbol{A}^*$. Additionally, because new exploration phases occur at powers of 2, the overall regret becomes upper bounded by $O(K(T)\log(T))$. The rigorous justification for this procedure is provided in the proof of Theorem 6.

Theorem 6 *If the players follow* `mCascadeDSEE` *in Algorithm 3 in the setting of Problem C, then we have the following regret bound:*

$$R_T \leq O(K(T)\log(T)) \tag{5}$$

The full proof is in section 4.2. Notice the lack of dependence on the gap $\Delta_{\boldsymbol{e},\boldsymbol{K}}$ in the above. This is because there is a constant $\sum_{t=1}^{\infty} t^{-2K_0(t)\epsilon^2}$ whose order depends on $\epsilon < \frac{1}{2}\min_{\boldsymbol{e},\boldsymbol{e}^*} \Delta_{\boldsymbol{e},\boldsymbol{e}^*} = \frac{1}{2}\Delta_{\boldsymbol{K}+\mathbf{1},\boldsymbol{K}}$.

Algorithm 3: `mCascadeDSEE`

```
1  Players will agree to an ordering of $\boldsymbol{E}$ held constant during the entire game.
2  Pick a monotonic function $K(\lambda) : \mathbb{N} \to \mathbb{N}$ such that $\lim_{t\to\infty} K(\lambda) = \infty$. First, let
     $\lambda = 1$.
3  for each joint-item $\boldsymbol{e} \in \boldsymbol{E}$ do
4      For $K(\lambda)$ rounds, each player recommends their individual items that make up
         the joint-action $\boldsymbol{A_t}$ starting with $\boldsymbol{e}$ followed by the next $K - 1$ joint-items after,
         according to the ordering.
5  end
6  for each player $i \in \{1, \dots, M\}$ do
7      Player $i$ calculates the empirical attraction probability for each joint-item $\boldsymbol{e}$.
8      Player $i$ selects the top $K$ joint-items with the highest-attraction probabilities,
         and in the event of a tie, selects arbitrarily. They arrange them into $\boldsymbol{A}_t$ by the
         agreed upon order, and commits to their corresponding items up until the next
         power of 2.
9  end
10 When $t = 2^n$ for some $n \geq \lfloor \log_2(K(1)L^M) \rfloor + 1$, go back to step (3) and start a new
     exploration phase, incrementing $\lambda$ by 1.
```

4 Regret Analysis

4.1 Problem B

Here we prepare the proof of the upper bound in Theorem 5. We first cite some important auxiliary definitions and results from previous literature, particularly Kveton et al. (2015a), attempting to parallel notation when possible.

- For any joint ranking $\boldsymbol{A}_t$, define the *permutation of optimal joint-items*, $\boldsymbol{\pi}_t$ as such: for $k = 1, \cdots, K$, if the k-th joint item in $\boldsymbol{A}_t$ is optimal, set $\boldsymbol{\pi}_t(k) = \boldsymbol{a}_k^t$. The remaining joint items in $\boldsymbol{A}_t$ are positioned arbitrarily.

- For any optimal joint-item $\boldsymbol{e}^*$ and sub-optimal joint-item $\boldsymbol{e}$, let $G_{\boldsymbol{e},\boldsymbol{e}^*,t}$ be the event $\boldsymbol{e}$ is chosen instead of item $\boldsymbol{e}^*$ at time t, *and that* $\boldsymbol{e}$ *is observed*. That is:

$$G_{\boldsymbol{e},\boldsymbol{e}^*,t} = \{\exists 1 \leq k \leq K \text{ s.t. } \boldsymbol{a}_k^t = \boldsymbol{e}, \boldsymbol{\pi}_t(k) = \boldsymbol{e}^*, \text{ and } \mathbf{w}_t(\boldsymbol{a}_1^t), \cdots, \mathbf{w}_t(\boldsymbol{a}_{k-1}^t) = 0\}$$

These definitions set up the use of Theorem 1 from Kveton et al. (2015a), which allows us to decompose the expected regret on round t by looking at the sub-optimality gaps and the G event for all suboptimal items.

Lemma 7 *(Theorem 1, Kveton et al. (2015a), adapted for joint-items and joint-rankings)*

$$\mathbb{E}_t\left[R(\boldsymbol{A}_t, \boldsymbol{w}_t)\right] \leq \sum_{\boldsymbol{e}=\boldsymbol{K}+\boldsymbol{1}}^{\boldsymbol{L}} \sum_{\boldsymbol{e}^*=\boldsymbol{1}}^{\boldsymbol{K}} \Delta_{\boldsymbol{e},\boldsymbol{e}^*} \mathbb{E}_t\left[\mathbb{1}\{G_{\boldsymbol{e},\boldsymbol{e}^*,t}\}\right]$$

We now prove Theorem 5.

Proof (Theorem 5) Define $\mathcal{E}_t^i = \{\exists \boldsymbol{e} \in \boldsymbol{E} \text{ s.t. } |\bar{w}(\boldsymbol{e}) - \hat{\mathbf{w}}_{n^i_{t-1}(\boldsymbol{e})}(\boldsymbol{e})| \geq c_{n^i_{t-1}(\boldsymbol{e})}\}$. Thus, $\bigcup_i \mathcal{E}_t^i$ is the event that for some player i, *there exists some joint-item* $\boldsymbol{e}$ where the $\bar{w}(e)$ is *not* within the player i's UCB interval around $\hat{\mathbf{w}}_{n^i_{t-1}(\boldsymbol{e})}(\boldsymbol{e})$. Let $\bigcap_i \bar{\mathcal{E}}_t^i$ be the complement; that is, for all players and for all joint items $\boldsymbol{e}$, $\bar{w}(\boldsymbol{e})$ lies within each player's respective UCB interval for $\boldsymbol{e}$. Decompose the regret of `mCascadeUCB-Intervals` as

$$R_T = \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\left\{\bigcup_i \mathcal{E}_t^i\right\} R(\boldsymbol{A}_t, \mathbf{w}_t)\right] + \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\left\{\bigcap_i \bar{\mathcal{E}}_t^i\right\} R(\boldsymbol{A}_t, \mathbf{w}_t)\right] \tag{6}$$

where $R(\boldsymbol{A}_t, \mathbf{w}_t)$ is the regret incurred on time t.

For the first term of 6, note:

$$\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\{\bigcup_i \mathcal{E}_t^i\} R(\boldsymbol{A}_t, \mathbf{w}_t)\right] \leq \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\left\{\bigcup_i \mathcal{E}_t^i\right\}\right] \leq \sum_{i=1}^{M} \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\left\{\mathcal{E}_t^i\right\}\right] \leq \frac{\pi^2}{3} M L^M$$

where the third inequality follows from 1) Theorem 2 from Kveton et al. (2015a) which uses the fact that our UCB intervals across all L^M joint-items were constructed to hold with high probability (Hoeffding's Inequality) and 2) the aforementioned upper bound holds for each player.

Next, we bound the number of observations of a suboptimal joint-item $\boldsymbol{e}$ under the good event $\bigcap_i \bar{\mathcal{E}}_t$ by any player. Since the players' clicks/observations are i.i.d. and regret is calculated in expectation, consider the regret incurred by an arbitrary player i and let $n^i_{t-1}(\boldsymbol{e})$ denote the number of observations for item $\boldsymbol{e}$ for player i up to time t. We first decompose the second term of 6 into the first $2L^M$ rounds, where $R(\boldsymbol{A}_t, \mathbf{w}_t) \leq 1$, and the remaining $T - 2L^M$ rounds. Invoking Lemma 7 for the latter, the second term of 6 is thus upper bounded as:

$$\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\left\{\bigcap_i \bar{\mathcal{E}}_t\right\} R(\boldsymbol{A}_t, \mathbf{w}_t)\right] \leq \sum_{\boldsymbol{e}=\boldsymbol{K}+\boldsymbol{1}}^{\boldsymbol{L}^{\boldsymbol{M}}} \mathbb{E}\left[\sum_{\boldsymbol{e}^*=\boldsymbol{1}}^{\boldsymbol{K}} \sum_{t=2L^M+1}^{T} \Delta_{\boldsymbol{e},\boldsymbol{e}^*} \mathbb{I}\left\{\bigcap_i \bar{\mathcal{E}}_t, G_{\boldsymbol{e},\boldsymbol{e}^*,t}\right\}\right] + 2L^M \tag{7}$$

where $G_{\boldsymbol{e},\boldsymbol{e}^*,t}$ is now the event that $\boldsymbol{e}$ is chosen instead of $\boldsymbol{e}^*$ at time t, and that $\boldsymbol{e}$ is observed for player i. Select any optimal joint-item $\boldsymbol{e}^*$. For any time $t > 2L^M$, under $G_{\boldsymbol{e},\boldsymbol{e}^*,t}$, this implies $\boldsymbol{e}$ and $\boldsymbol{e}^*$'s UCB intervals are *not* disjoint for all players (otherwise $\boldsymbol{e}$ would've been eliminated), which means $\hat{\mathbf{w}}_{n^i_{t-1}(\boldsymbol{e})}(\boldsymbol{e}) + c_{n^i_{t-1}(\boldsymbol{e})} \geq \hat{\mathbf{w}}_{n^i_{t-1}(\boldsymbol{e}^*)}(\boldsymbol{e}^*) - c_{n^i_{t-1}(\boldsymbol{e}^*)}$. Additionally, under $\bigcap_i \bar{\mathcal{E}}_t$, we know the true attraction probabilities lie within each player's respective UCB intervals, i.e. $|\bar{w}(\boldsymbol{e}) - \hat{\mathbf{w}}_{n^i_{t-1}(\boldsymbol{e})}(\boldsymbol{e})| < c_{n^i_{t-1}(\boldsymbol{e})}$ and $|\bar{w}(\boldsymbol{e}^*) - \hat{\mathbf{w}}_{n^i_{t-1}(\boldsymbol{e}^*)}(\boldsymbol{e}^*)| < c_{n^i_{t-1}(\boldsymbol{e}^*)}$. With these facts together, it holds that:

$$\bar{w}(\boldsymbol{e}) + 2c_{n^i_{t-1}(\boldsymbol{e})} \geq \bar{w}(\boldsymbol{e}^*) - 2c_{n^i_{t-1}(\boldsymbol{e}^*)}$$

which implies:

$$\Delta_{\boldsymbol{e},\boldsymbol{e}^*} \leq 2(c_{n^i_{t-1}(\boldsymbol{e})} + c_{n^i_{t-1}(\boldsymbol{e}^*)})$$

$$\leq 2\left(\sqrt{\frac{1.5\log T}{n^i_{t-1}(\boldsymbol{e})}} + \sqrt{\frac{1.5\log T}{n^i_{t-1}(\boldsymbol{e}^*)}}\right) \qquad \text{(By definition of } c\text{)}$$

By Lemma 8, for any $t > 2L^M$, we have $n^i_{t-1}(\boldsymbol{e}^*) \geq \frac{n^i_{t-1}(\boldsymbol{e})}{4K}$. Thus,

$$\Delta_{\boldsymbol{e},\boldsymbol{e}^*} \leq 2\left(\sqrt{\frac{1.5\log T}{n^i_{t-1}(\boldsymbol{e})}} + \sqrt{\frac{6K\log T}{n^i_{t-1}(\boldsymbol{e})}}\right) \qquad \text{(by Lemma 8)}$$

$$\implies n^i_{t-1}(\boldsymbol{e}) \leq \frac{(6+24K+24\sqrt{K})\log T}{\Delta^2_{\boldsymbol{e},\boldsymbol{e}^*}}$$

Let $\tau_{\boldsymbol{e},\boldsymbol{e}^*} = \frac{(6+24K+24\sqrt{K})\log T}{\Delta^2_{\boldsymbol{e},\boldsymbol{e}^*}}$. Therefore,

$$\sum_{\boldsymbol{e^*=1}}^{\boldsymbol{K}} \sum_{t=2L^M+1}^{T} \Delta_{\boldsymbol{e},\boldsymbol{e}^*}\mathbb{I}\left\{\bigcap_i \bar{\mathcal{E}}_t, G_{\boldsymbol{e},\boldsymbol{e}^*,t}\right\} \leq \sum_{\boldsymbol{e^*=1}}^{\boldsymbol{K}} \Delta_{\boldsymbol{e},\boldsymbol{e}^*} \sum_{t=2L^M+1}^{T} \mathbb{I}\left\{\bigcap_i \{n^i_{t-1}(e) \leq \tau_{\boldsymbol{e},\boldsymbol{e}^*}\}, G_{\boldsymbol{e},\boldsymbol{e}^*,t}\right\}. \tag{8}$$

Let:

$$\mathbf{M}_{\boldsymbol{e},\boldsymbol{e}^*} = \sum_{t=2L^M+1}^{T} \mathbb{I}\{\bigcap_i \{n^i_{t-1}(e) \leq \tau_{\boldsymbol{e},\boldsymbol{e}^*}\}, G_{\boldsymbol{e},\boldsymbol{e}^*,t}\}$$

be the inner sum in 8. Now note that 1) across all players i, the number of observations $n^i_{t-1}(\boldsymbol{e})$ of item $\boldsymbol{e}$ increases by one when the event $G_{\boldsymbol{e},\boldsymbol{e}^*,t}$ happens for that player for any optimal item $\boldsymbol{e}^*$, 2) for any given player, the event $G_{\boldsymbol{e},\boldsymbol{e}^*,t}$ happens for at most one optimal $\boldsymbol{e}^*$ at any time t, and 3) $\tau_{\boldsymbol{e},\boldsymbol{1}} \leq \cdots \leq \tau_{\boldsymbol{e},\boldsymbol{K}}$.

From these facts, we can bound the number of times the indicator event is true over $t = 2L^M, \cdots, T$ as $\mathbf{M}_{\boldsymbol{e},\boldsymbol{e}^*} \leq \tau_{\boldsymbol{e},\boldsymbol{e}^*}$, and moreover, $\sum_{\boldsymbol{e^*=1}}^{\boldsymbol{K}} \mathbf{M}_{\boldsymbol{e},\boldsymbol{e}^*} \leq \tau_{\boldsymbol{e},\boldsymbol{K}}$. Thus, the right-hand side of 8 is bounded above by:

$$\max\left\{\sum_{\boldsymbol{e^*=1}}^{\boldsymbol{K}} \Delta_{\boldsymbol{e},\boldsymbol{e}^*} m_{\boldsymbol{e},\boldsymbol{e}^*} : 0 \leq m_{\boldsymbol{e},\boldsymbol{e}^*} \leq \tau_{\boldsymbol{e},\boldsymbol{e^*}}, \sum_{\boldsymbol{e^*=1}}^{\boldsymbol{K}} m_{\boldsymbol{e},\boldsymbol{e}^*} \leq \tau_{\boldsymbol{e},\boldsymbol{K}}\right\}.$$

Since the gaps are decreasing, $\Delta_{\boldsymbol{e},\boldsymbol{1}} \geq \cdots \geq \Delta_{\boldsymbol{e},\boldsymbol{K}}$, we maximize the quantity above by making $m_{\boldsymbol{e},\boldsymbol{e}^*}$ the largest possible for lower-numbered optimal joint-items, i.e. $m^*_{\boldsymbol{e},\boldsymbol{1}} = \tau_{\boldsymbol{e},\boldsymbol{1}}, m^*_{\boldsymbol{e},\boldsymbol{2}} = \tau_{\boldsymbol{e},\boldsymbol{2}} - \tau_{\boldsymbol{e},\boldsymbol{1}}, \ldots, m^*_{\boldsymbol{e},\boldsymbol{K}} = \tau_{\boldsymbol{e},\boldsymbol{K}} - \tau_{\boldsymbol{e},\boldsymbol{K-1}}$ (where the first constraint is satisfied by fact 3). Substituting in for $\tau_{\boldsymbol{e},\boldsymbol{e^*}}$, we see 8 is bounded above by:

$$\left[\Delta_{\boldsymbol{e},\boldsymbol{1}}\frac{1}{\Delta^2_{\boldsymbol{e},\boldsymbol{1}}} + \sum_{\boldsymbol{e^*=2}}^{\boldsymbol{K}} \Delta_{\boldsymbol{e},\boldsymbol{e}^*}\left(\frac{1}{\Delta^2_{\boldsymbol{e},\boldsymbol{e}^*}} - \frac{1}{\Delta^2_{\boldsymbol{e},\boldsymbol{e^*-1}}}\right)\right](6+24K+24\sqrt{K})\log T. \tag{9}$$

By Lemma 3 of Kveton et al. (2014), the term inside the square brackets of 9 is bounded above by $\frac{2}{\Delta_{\boldsymbol{e},\boldsymbol{K}}}$. Thus, summing over all suboptimal joint-items $\boldsymbol{e}$, we get that the first term of the right-hand side of 7 is bounded above as:

$$\sum_{\boldsymbol{e}=\boldsymbol{K}+\boldsymbol{1}}^{\boldsymbol{L^M}} \frac{12+48K+48\sqrt{K}}{\Delta_{\boldsymbol{e},\boldsymbol{K}}} \log T$$

Chaining all inequalities finishes the proof. ■

Lemma 8 *Suppose the players are following* `mCascadeUCB-Intervals`*. Let $\boldsymbol{e}$ be any suboptimal item and $\boldsymbol{e}^*$ be any optimal item in a ground set $\boldsymbol{E}$ with cardinality L. Then for any $t > 2L^M$, if $\bigcap_i \bar{\mathcal{E}}_t$ happens, the number of observations by player i of optimal item $\boldsymbol{e}^*$ is lower bounded as:*

$$n^i_{t-1}(\boldsymbol{e}^*) \geq \frac{1}{4K} n^i_{t-1}(\boldsymbol{e})$$

Proof As we are in $\bigcap_i \bar{\mathcal{E}}_t$, we can consider the case where both $\boldsymbol{e}^*$ and $\boldsymbol{e}$ are both still within the desired set at time t; the case when $\boldsymbol{e}$ has been eliminated follows as a result because $\boldsymbol{e}$ no longer will be observed and thus the lower bound is unchanging. Without loss of generality, assume the size of the desired set at time t is L^M. Partition the t rounds into groups of size L^M, namely:

$$\{1, ..., t\} = \{1, ..., L^M\} \cup \{L^M + 1, ..., 2L^M\} \cup \cdots \cup \{\lfloor \frac{t}{L^M} \rfloor L^M + 1, ..., t\}.$$

In the event the desired set is smaller than L^M, simply partition the t rounds by each loop over the current desired set. Since by `mCascadeUCB-Intervals`, the arms are recommended cyclically, in each group of rounds (e.g. $\{nL^M + 1, ..., (n+1)L^M\}$), for exactly one round, $\boldsymbol{e^*}$ must be the first joint-item to be recommended and therefore will be observed by all the users for that round. Therefore, $n_{t-1}(\boldsymbol{e}) \geq 1$ (which is true for $t > L^M$),

$$n^i_{t-1}(\boldsymbol{e^*}) \geq \lfloor \frac{t}{L^M} \rfloor \geq \frac{t}{2L^M}$$

where the last inequality holds as $\frac{t}{L^M} \geq 2$. Furthermore, by `mCascadeUCB-Intervals`, $\boldsymbol{e}$ is recommended (and therefore observed) *at most* K times over every round. Therefore, we have whenever $n_{t-1}(\boldsymbol{e}) \geq 1$ (again true for $t > L^M$),

$$n^i_{t-1}(\boldsymbol{e}) \leq K \lceil \frac{t}{L^M} \rceil \leq 2K \frac{t}{L^M}$$

where the last inequality holds as $\frac{t}{L^M} \geq 1$. Combining both inequalities above, we have,

$$n^i_{t-1}(\boldsymbol{e}^*) \geq \frac{1}{4K} n^i_{t-1}(\boldsymbol{e}).$$

■

4.2 Problem C

In this section, we provide the proof of Theorem 6. It follows many of the same ideas in Chang et al. (2022).

Proof (Theorem 6) Decompose the total regret $R_T = R_{T,E} + R_{T,C}$, where $R_{T,E}$ is the regret incurred during the exploration sequence spaced at powers of 2, while $R_{T,C}$ is the regret incurred during the commitment phase when committing to the ranking of the top K items with the highest estimated means.

During the λ-th exploration phase, each joint-item $\boldsymbol{e}$ is ranked first $K(\lambda)$ times. Let $\nu_t(\boldsymbol{e})$ denote the total number of times joint-item $\boldsymbol{e}$ is ranked first during exploration up to time t. The number of exploration phases is approximately $\log_2(T)$, since exploration occurs at intervals based on powers of 2. It follows that $\nu_t(\boldsymbol{e}) \leq K(\lfloor \log_2(t) \rfloor)\lceil \log_2(t) \rceil$.

As $R_t(\boldsymbol{A_t}, \mathbf{w_t}) \leq 1$, regret incurred during exploration is thus bounded by the total number of exploration rounds, i.e.:

$$R_{T,E} \leq \sum_{\boldsymbol{e} \in \boldsymbol{E}} \nu_t(\boldsymbol{e}) \leq L^M \cdot K(\lfloor \log_2(T) \rfloor) \cdot \lceil \log_2(T) \rceil.$$

Next, we show $R_{T,C}$ is bounded by a constant less than ∞. Let $\epsilon < \frac{1}{2}\Delta_{\boldsymbol{K+1},\boldsymbol{K}}$, where $\boldsymbol{K}$ is the optimal item with the lowest true attraction probability and $\boldsymbol{K+1}$ is the suboptimal item with the highest true attraction probability (thus, $\Delta_{\boldsymbol{K+1},\boldsymbol{K}}$ is the minimum sub-optimality gap). Consider the good event $G^i_t(\boldsymbol{e})$ where for player i at time t, the empirical probability for item $\boldsymbol{e}$ ($\hat{\mathbf{w}}_{n^i_{t-1}(\boldsymbol{e})}(\boldsymbol{e})$) is within ϵ of the true probability for item e ($\bar{w}(\boldsymbol{e})$), i.e.:

$$G^i_t(\boldsymbol{e}) = \{|\hat{\mathbf{w}}_{n^i_{t-1}}(\boldsymbol{e}) - \bar{w}(\boldsymbol{e})| < \epsilon\}$$

It follows that when $G_t = \bigcap_{\boldsymbol{e},i} G^i_t(\boldsymbol{e})$ all players collectively choose $\boldsymbol{A}^* = (\boldsymbol{1}, \ldots, \boldsymbol{K})$ and no regret is incurred. Using the law of total expectation, our commit-phase regret can thus be bounded as:

$$
\begin{aligned}
R_{T,C} &= \sum_{t \in C} \mathbb{E}[R_t \mid G_t] P(G_t) + \mathbb{E}[R_t \mid G_t^c] P(G_t^c) \\
&\leq \sum_{t=1}^{T} P(G_t^c) \quad (\text{since } \mathbb{E}[R_t \mid G_t^c] \leq 1 \text{ and } \mathbb{E}[R_t \mid G_t] = 0) \\
&\leq \sum_{t=1}^{T} \sum_{i=1}^{M} \sum_{\boldsymbol{e} \in \boldsymbol{E}} P\left(G_t^i(\boldsymbol{e})^c\right) \quad (\text{applying De Morgan's Law and union bound}) \\
&= M \sum_{t=1}^{T} \sum_{\boldsymbol{e} \in \boldsymbol{E}} P\left(\left|\hat{\mathbf{w}}_{n_{t-1}(\boldsymbol{e})}(\boldsymbol{e}) - \bar{w}(\boldsymbol{e})\right| \geq \epsilon\right) \quad (\text{by i.i.d. rewards}) \\
&\leq 2M \sum_{t=1}^{T} \sum_{\boldsymbol{e} \in \boldsymbol{E}} \exp\left(-2n_{t-1}(\boldsymbol{e})\epsilon^2\right) \quad (\text{by Hoeffding's inequality}) \\
&\leq 2M \sum_{t=1}^{T} \sum_{\boldsymbol{e} \in \boldsymbol{E}} \exp\left(-2K_0(t)\log_2(t)\epsilon^2\right) \quad (\text{using } n_{t-1}(\boldsymbol{e}) \geq K_0(t)\log_2(t)) \\
&\leq 2ML^M \sum_{t=1}^{T} t^{-2K_0(t)\epsilon^2} \\
&\leq 2ML^M \sum_{t=1}^{\infty} t^{-2K_0(t)\epsilon^2}
\end{aligned}
$$

where $n_{t-1}(\boldsymbol{e}) \geq K_0(t)\log_2(t)$ follows from Claim 2, Theorem 5 in Chang et al. (2022)). The function $f(t) = t^{-2K_0(t)\epsilon^2}$ is monotonically decreasing for $t \geq 1$ since $K_0(t)$ is a non-decreasing function that tends to infinity as $t \to \infty$. Thus, the sum $\sum_{t=1}^{\infty} t^{-2K_0(t)\epsilon^2}$ can be bounded by:

$$
\sum_{t=1}^{\infty} t^{-2K_0(t)\epsilon^2} \leq 1 + \int_{1}^{\infty} t^{-2K_0(t)\epsilon^2}\, dt
$$

Since $K_0(t)$ tends to infinity, there exists an integer N such that for all $t > N$, $2K_0(t)\epsilon^2 \geq 3$ so that $t^{-2K_0(t)\epsilon^2} \leq t^{-3}$. As $\gamma > 1$ ensures that the integral $\int_1^\infty t^{-\gamma}\, dt$ converges, it follows that $\int_N^\infty t^{-2K_0(t)\epsilon^2} dt < \infty$. Therefore, $R_{T,C}$ is bounded by a constant, leading to the total regret bound:

$$
R_T \leq O(K(T)\log(T)).
$$

■

5 Experiments

In this section, we demonstrate the effectiveness of our algorithms with experiments. We run our experiments with $L = 5$ individual arms per players, $K = 10$ recommended positions, and $M = 4$ players. Note that this gives rise to $5^4 = 625$ total joint actions. We run for

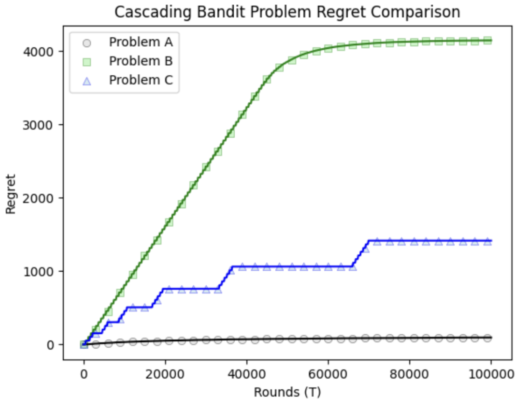

Figure 1: Experiments with $L = 5$ individual arms per players, $K = 10$ recommended positions, and $M = 4$ players. Our regret bounds are in 1. We run for $T = 100,000$ rounds and average the regret across 10 rounds. The black curve represents `mCascadeUCB` for Problem A (information asymmetry in actions) and is the benchmark. The green curve represents `mCascade-Intervals` for Problem B (information asymmetry in rewards). The blue curve represents `mCascadeDSEE` for problem C (information asymmetry in both).

$T = 100,000$ rounds and average the regret across 10 rounds. Our regret curves are in Figure 1.

The black curve represents `mCascadeUCB` for Problem A (information asymmetry in actions) and is the benchmark. Note in this plot that the algorithm for Problem A has the same regret bound as Kveton et al. (2015a) which will essentially serve as our benchmark for our algorithms.

The green curve represents `mCascade-Intervals` for Problem B (information asymmetry in rewards). Notice that for this problem the regret curve is eventually flat. This is because all the suboptimal joint items eventually get eliminated, and thus the players only pull recommend the top K items (this algorithm is useful for best arm identification as well in practice). The reason this performs worse than problem C despite having a stronger regret bond is because of the time it takes to remove an arm from the desired set. This can be sped up by decreasing the length of the interval by a constant factor.

The blue curve represents `mCascadeDSEE` for problem C (information asymmetry in both). Note that for this algorithm, initially the regret is linear since the players do not have enough samples to commit to the best top K items. Eventually, the graph looks like a "staircase". This is because in the exploration phases, the regret is linear and thus the plot is steep in those areas. In the committing phases, the plot is flat because the players are committing to the best actions. Furthermore, the flat parts are growing because the exploration phases are occurring at powers of 2. As T grows larger this cuve will eventually do worse than the green curve (Problem B regret curve).

6 Conclusion

In this paper, we presented a novel multi-agent extension of cascading bandits, featuring joint items and rankings requiring coordination between decentralized players. We presented algorithms for handling three variants of information asymmetry in the cascading setting, proved upper regret bounds, and demonstrated their performance with experiments: `mCascadeUCB` achieves $O(\log T)$ gap-dependent regret in the action-asymmetric setting, `mCascadeUCB-Intervals` achieves the same in the reward-asymmetric setting, and `mCascadeDSEE` achieves nearly log regret in the action and reward-asymmetric setting.

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